

On a Problem of Projectiles

DISSERTATION

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF
THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY
WITH THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

BY

ALLAN WILSON HOBBS

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ON A PROBLEM OF PROJECTILES

BY A. W. HOBBS

The purpose of this paper is to show how the solution of certain differential equations connected with the motion of a particle can be obtained, and then to discuss the integral curves arising from these solutions.

Two cases are considered, first that in which the particle is moving in the earth's gravitational field, and second that in which the force varies directly as the distance.

It was hoped at the beginning of this work that the same methods would be found applicable to the case of the inverse square law of force but such was not found to be true, as the equations for that case do not yield to the methods of solution used in the other two cases.

While this paper does not deal with the practical problem of gun fire it does deal with the theory which underlies that problem hence it has been called A Problem of Projectiles.

THE GRAVITATIONAL CASE

If a particle be projected from a point A whose coordinates are (x_0, y_0) , with initial velocity v and at an angle of elevation α , the equations of motion are

$$(1) \quad \begin{aligned} x - x_0 &= vt \cos \alpha, \\ y - y_0 &= vt \sin \alpha - \frac{1}{2}gt^2. \end{aligned}$$

The conditions necessary to send the particle through another point B , which we shall take as the origin, are obtained by putting $x = y = 0$ in equations (1).

Doing this and eliminating t we have

$$(2) \quad y_0 = x_0 \tan \alpha + (g/2v^2)(1 + \tan^2 \alpha)x_0^2,$$

which is the equation of the locus of all points whence the particle can be projected with the given velocity and at the given angle so as to pass through B .

Considering now the angle α as variable we may put

$$\tan \alpha = dy/dx.$$

Equation (2) then becomes, after dropping subscripts and replacing $g/2v^2$ by $1/4h$,

$$(3) \quad 4h \left(y - x \frac{dy}{dx} \right) - x^2 \left[1 + \left(\frac{dy}{dx} \right)^2 \right] = 0.$$

This differential equation gives for any point the two directions of projection with the velocity v in order to hit point B . The usual methods for solving differential equations of the second degree are not practicable here but if we change to a special type of line coordinates the solution is readily obtained.

We take

$$(4) \quad \begin{aligned} x \cos \omega + y \sin \omega &= p, \\ -x \sin \omega + y \cos \omega &= p'. \end{aligned}$$

Solving these for x and y we get

$$(5) \quad \begin{aligned} x &= p \cos \omega - p' \sin \omega, \\ y &= p' \cos \omega + p \sin \omega. \end{aligned}$$

From these we find

$$\frac{dy}{dx} = -\cot \omega.$$

Putting these values of x , y and dy/dx in equation (3) we have

$$p' \sin \omega = p \cos \omega \pm \sqrt{4hp \sin \omega}.$$

In this equation the variables are separated by putting

$$p = u^2 \sin \omega.$$

We then have

$$\sin \omega du = \sqrt{h} d\omega.$$

Integrating and eliminating u we get for the family of integral curves

$$(6) \quad p = h \sin \omega \left[\log \left(a \tan \frac{\omega}{2} \right) \right]^2.$$

The tangents to the two members of this family which pass

through any point give the directions for firing with the given velocity so as to hit the point B . The directions coincide for points which satisfy the discriminant relation of equation (3), namely

$$(7) \quad x^2 - 4hy - 4h^2 = 0.$$

This is the equation of a parabola with the origin at the focus. The single direction of firing for a point on it is along the normal at the point, since we know that the two paths from this point to the point B coincide in this case and that the focus of the trajectory is therefore on the line joining this point to the point B . Thus the normal to the parabola (7) at any point on it is tangent to the trajectory for that point.

From equations (5) we see that

$$\omega = \alpha - \frac{\pi}{2},$$

and that $dp/d\omega$ gives the distance from the foot of the perpendicular from the origin to a tangent to the point of contact of the tangent. We shall consider this quantity positive when p and ω are both increasing or both decreasing.

By a comparison of the radii of curvature of the integral curves and the path curves we can find their relative positions, which might be of some interest. The radius of curvature in terms of coordinates p , ω is $R = p + p''$.

The radius of curvature for

$$p = h \sin \omega \left[\log \left(a \tan \frac{\omega}{2} \right) \right]^2,$$

is

$$R_1 = 2h \cot \omega \log \left(a \tan \frac{\omega}{2} \right) + 2h \csc \omega.$$

The equation of the path curve from x_0 , y_0 is

$$y - y_0 = (x - x_0) \tan \alpha - (g/2v^2)(x - x_0)^2(1 + \tan^2 \alpha),$$

and the radius of curvature at the point $x = x_0$, $y = y_0$ is

$$R_2 = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} = 2h \csc \omega.$$

The difference between this and the radius of curvature of the integral curve through x_0, y_0 is given by

$$R_1 - R_2 = 2h \cot \omega \log \left(a \tan \frac{\omega}{2} \right).$$

(Of the two integral curves through any point we shall consider only the one which gives the lower angle.)

The parabola of equation (7) divides the plane into points outside, from which it is not possible to reach the point B and points inside from which there are two directions of firing to reach B . Whatever can be said for points inside on the right of the axis can be made to apply to points on the left by replacing α by $\pi - \alpha$. For points on the right α cannot be less than $\pi/2$. For α between $\pi/2$ and π ,

$$R_1 - R_2 = 2h \cot \omega \log \left(a \tan \frac{\omega}{2} \right)$$

is negative since ω is between 0 and $\pi/2$, consequently R_2 is greater than R_1 and we see that at such points the path curves lie below the integral curves.

For α between π and $3\pi/2$, $R_1 - R_2$ is positive so that the integral curves lie above the path curves. The case $\alpha = \pi$, or $\omega = \pi/2$, can be shown to belong to the latter class by finding the angle at which the path curve arrives at the point B .

From the above considerations we see that for points for which the direction of projection is above the horizontal the path of the particle is entirely above the integral curve, while at the points for which the direction is below the horizontal the path is entirely below the integral curve. Thus if a hill were constructed in the shape of one of these integral curves a shot fired along the tangent would reach the target B only from points beyond the top of the hill from B , since a shot fired along the tangent from a point on the side next to B would pass beneath the surface. The same thing would hold for shots fired from B towards A . Thus with direct fire B could not hit A , while A could hit B if situated on the other side of the hill from him.

The maximum points of the curves of the family

$$p = h \sin \omega \left[\log \left(a \tan \frac{\omega}{2} \right) \right]^2$$

separate the points of the plane from which the path curves lie above the integral curves from those from which the path curves lie below the integral curves. We have seen that these maximum points themselves belong to the latter class. They are all situated on the parabola

$$x^2 = 4hy,$$

as is seen from the fact that the horizontal tangents are found by putting $\omega = \pi/2$ in the family of integral curves. This gives for the vertical distance of a maximum point

$$p = y = h(\log a)^2.$$

The horizontal distance as found from $dp/d\omega$ for $\omega = \pi/2$ is

$$\frac{dp}{d\omega} = x = 2h(\log a)^2,$$

and such values of x and y evidently lie on the parabola

$$x^2 = 4hy.$$

To construct the integral curves we can make use of the following considerations. We should expect these curves to have cusps at the points on the bounding parabola, and since the radius of curvature at a cusp is to vanish we see that

$$\log \left(a \tan \frac{\omega}{2} \right) = -\sec \omega$$

is the condition for a cusp.

If we put this value in the equation of the family of integral curves we get the envelope of the cusp tangents

$$p = h \sec \omega \tan \omega,$$

but this is the equation of the evolute of the parabola

$$x^2 - 4hy - 4h^2 = 0.$$

To be sure that the cusp points are actually on this parabola we consider the value of $dp/d\omega$,

$$\frac{dp}{d\omega} = h \cos \omega \left[\log \left(a \tan \frac{\omega}{2} \right) \right]^2 + 2h \log \tan \left(a \tan \frac{\omega}{2} \right),$$

and putting $-\sec \omega$ for $\log (a \tan \omega/2)$, we get

$$p = -h \sec \omega,$$

which, if we use the same ω that we have for the evolute, is simply the distance from the origin to that tangent of the parabola whose point of contact is the point in which the normal we are considering meets the parabola, and this shows that the cusps are on the given parabola. The same thing can be seen from the equations

$$\begin{aligned} x &= p \cos \omega - p' \sin \omega, \\ y &= p' \cos \omega + p \sin \omega, \end{aligned}$$

for if in these equations we put

$$\begin{aligned} p &= h \sin \omega \sec^2 \omega, \\ p' &= -\sec \omega, \end{aligned}$$

and eliminate ω we obtain again the equation of the bounding parabola

$$x^2 - 4hy - 4h^2 = 0.$$

The values of ω giving lines common to a definite integral curve and the evolute are found by equating the right members in

$$p = h \sin \omega \left[\log \left(a \tan \frac{\omega}{2} \right) \right]^2,$$

and

$$p = h \sec \omega \tan \omega,$$

which gives

$$\log \left(a \tan \frac{\omega}{2} \right) = \pm \sec \omega.$$

Approximate solutions are found by putting

$$\tan \frac{\omega}{2} = t,$$

so that

$$\log at = \pm \frac{1+t^2}{1-t^2}.$$

From this relation common values of t can be calculated and as many lines of the curve drawn as we wish. Figure 1 shows a few of these curves drawn for $h = 1$. Figure 2 shows a single curve drawn to a larger scale with path curves for different points on it.



FIG. 1.



FIG. 2.

THE DIRECT DISTANCE CASE

If a particle be projected with velocity v and angle α in a field where the force varies directly as the distance the equations of motion are

$$(1) \quad \begin{aligned} \frac{d^2x}{dt^2} &= -\mu x, \\ \frac{d^2y}{dt^2} &= -\mu y. \end{aligned}$$

Integrating we get

$$(2) \quad \begin{aligned} x &= A_1 \cos \sqrt{\mu} t + A_2 \sin \sqrt{\mu} t, \\ y &= B_1 \cos \sqrt{\mu} t + B_2 \sin \sqrt{\mu} t. \end{aligned}$$

Eliminating t we have as the equation of the path

$$(3) \quad (A_1 y - B_1 x)^2 + (A_2 y - B_2 x)^2 = (A_1 B_2 - A_2 B_1)^2.$$

We ask that this path shall pass through a fixed point say $(1, 0)$. The condition for this is

$$(4) \quad B_1^2 + B_2^2 = (A_1 B_2 - A_2 B_1)^2.$$

A_1, A_2, B_1, B_2 depend upon the initial conditions. Thus for $t = 0$,

$$A_1 = x_0, \quad B_1 = y_0, \quad A_2 = \frac{v \cos \alpha}{\sqrt{\mu}}, \quad B_2 = \frac{v \sin \alpha}{\sqrt{\mu}}.$$

Putting these values in (4) we get

$$(5) \quad y_0^2 - \frac{v^2 \sin^2 \alpha}{\mu} = \left(x_0 v \frac{\sin \alpha}{\sqrt{\mu}} - \frac{y_0 v \cos \alpha}{\sqrt{\mu}} \right)^2.$$

This equation represents the locus of points whence the particle can be projected at the given angle and with the given velocity so as to pass through the given point. As in the Gravitational Case we now consider α variable and put $\tan \alpha = dy/dx$.

Dropping subscripts and letting h represent v^2/μ we obtain the differential equation of the second degree

$$(6) \quad (hx^2 - y^2 - h) \left(\frac{dy}{dx} \right)^2 - 2hxy \frac{dy}{dx} + (h - 1)y^2 = 0,$$

which gives the two directions for any given point. These two directions will coincide at points satisfying the discriminant relation of equation (6), namely

$$(7) \quad \begin{aligned} h^2 x^2 y^2 &= (h - 1)(hx^2 - y^2 - h)y^2, \quad \text{or} \\ y^2 &= 0 \quad \text{and} \quad \frac{x^2}{1 - h} + \frac{y^2}{-h} = 1. \end{aligned}$$

Equation (6) can be solved exactly as in the former case. We

change to line coordinates as before by putting

$$\begin{aligned}x &= p \cos \omega - p' \sin \omega, \\y &= p' \cos \omega + p \sin \omega, \\ \frac{dy}{dx} &= -\cot \omega.\end{aligned}$$

The reduced equation is

$$p' \cos \omega + p \sin \omega = \sqrt{h} \sqrt{p^2 - \cos^2 \omega},$$

which by putting $u \cos \omega = p$ is reduced to

$$\frac{du}{\sqrt{u^2 - 1}} = \sqrt{h} \frac{d\omega}{\cos \omega}.$$

Integrating this we get

$$u + \sqrt{u^2 - 1} = c(\sec \omega + \tan \omega)^{\sqrt{h}}.$$

Eliminating u

$$p + \sqrt{p^2 - \cos^2 \omega} = c \cos \omega (\sec \omega + \tan \omega)^{\sqrt{h}},$$

also

$$\begin{aligned}p - \sqrt{p^2 - \cos^2 \omega} &= \frac{1}{c} \cos \omega (\sec \omega - \tan \omega)^{\sqrt{h}}, \\ p &= \frac{\cos \omega}{2} \left[c(\sec \omega + \tan \omega)^{\sqrt{h}} + \frac{1}{c} (\sec \omega - \tan \omega)^{\sqrt{h}} \right], \\ p' &= \frac{\sqrt{h} - \sin \omega}{2} [c(\sec \omega + \tan \omega)^{\sqrt{h}} \\ &\quad - \frac{\sqrt{h} + \sin \omega}{2} \left[\frac{1}{c} (\sec \omega - \tan \omega)^{\sqrt{h}} \right]].\end{aligned}$$

Going back to rectangular coordinates through the equations

$$\begin{aligned}x &= p \cos \omega - p' \sin \omega, \\ y &= p' \cos \omega + p \sin \omega,\end{aligned}$$

we find as the parametric equations of the family of integral curves

$$\begin{aligned}x &= \frac{1 - \sqrt{h} \sin \omega}{2} [c(\sec \omega + \tan \omega)^{\sqrt{h}} \\ &\quad + \frac{1 + \sqrt{h} \sin \omega}{2} \left[\frac{1}{c} (\sec \omega - \tan \omega)^{\sqrt{h}} \right],\end{aligned}$$

$$y = \frac{\sqrt{h} \cos \omega}{2} \left[c(\sec \omega + \tan \omega)^{\sqrt{h}} - \frac{1}{c}(\sec \omega - \tan \omega)^{\sqrt{h}} \right].$$

The discriminant relation (7) consists of the x -axis and the conic

$$\frac{x^2}{1-h} + \frac{y^2}{-h} = 1,$$

which is an hyperbola for h less than 1, and an ellipse with no real points for h greater than 1.

SPECIAL CASES

$$h = 0.$$

Here the discriminant reduces to $y = 0$ and the integral curves are given by

$$x = \frac{1}{2} \left(c + \frac{1}{c} \right), \quad y = 0.$$

Thus the integral curves are the various points on the x -axis, and since the velocity is zero we have the ordinary simple harmonic motion.

$$h = 1.$$

For this case the discriminant relation is given by $x = 0$ and $y^2 = 0$ and the integral curves by

$$\begin{aligned} x &= \frac{1 - \sin \omega}{2} [c(\sec \omega + \tan \omega)] \\ &\quad + \frac{1 + \sin \omega}{2} \left[\frac{1}{c}(\sec \omega - \tan \omega) \right], \\ y &= \frac{\cos \omega}{2} [c(\sec \omega + \tan \omega)] + \frac{1}{c} [\sec \omega - \tan \omega]. \end{aligned}$$

Eliminating ω we get the family of circles

$$c(x^2 + y^2) - (c^2 - 1)y - c = 0,$$

which pass through the two points $(1, 0)$ and $(-1, 0)$, having their centers on the y -axis. We should expect that through any point of the plane would pass two members of the family the tangents to which would give the two directions of projec-

tion for that point. In this case however we see that there is only one member of the family passing through the two fixed points and any third point. It appears from this that for this particular value of the velocity ($v^2 = \mu$) we have only one direction at a point to send the particle through the given fixed point. To explain this apparent loss of one of the directions we can return to equation (5) which for $h = 1$ becomes

$$y_0^2 \sin^2 \alpha + \sin^2 \alpha = x_0^2 \sin^2 \alpha - 2x_0 y_0 \sin \alpha \cos \alpha,$$

and which is satisfied for $\alpha = 0$ or $\alpha = \pi$. Throwing out the factor $\sin \alpha$ we see that the differential equation now reduces to one of the first degree. One direction is thus along the horizontal, the other being given by the one circle through the point.

$$h = \frac{1}{4}.$$

The discriminant relation here is

$$\frac{x^2}{\frac{3}{4}} - \frac{y^2}{\frac{1}{4}} = 1,$$

and the integral curves are

$$\begin{aligned} x &= \frac{2 - \sin \omega}{4} [c(\sec \omega + \tan \omega)^{1/2}] \\ &\quad + \frac{2 + \sin \omega}{4} \left[\frac{1}{c} (\sec \omega - \tan \omega)^{1/2} \right], \\ y &= \frac{\cos \omega}{4} \left[(c \sec \omega + \tan \omega)^{1/2} - \frac{1}{c} (\sec \omega - \tan \omega)^{1/2} \right]. \end{aligned}$$

If we put

$$\sec \omega + \tan \omega = t^2,$$

we find the family of rational sextics whose parametric equations are

$$\begin{aligned} x &= \frac{c^2 t^6 + 3t^4 + 3c^2 t^2 + 1}{4ct(t^4 + 1)}, \\ y &= \frac{2c^2 t^4 + 2t^2}{4ct(t^4 + 1)}. \end{aligned}$$

It is also evident that the integral curves are rational for all rational values of $\sqrt{h}$.

In the case of the direct distance as in the gravitational case the direction of projection at points on the curve given by the discriminant relation is along the normal, for since the two directions coincide at such points the one direction is given by the repeated root of

$$(hx^2 - y^2 - h) \left(\frac{dy}{dx} \right)^2 - 2hxy \left(\frac{dy}{dx} \right) + (h - 1)y^2 = 0,$$

which is

$$\frac{hxy}{hx^2 - y^2 - h}.$$

The slope of the normal to

$$\frac{x^2}{1 - h} - \frac{y^2}{h} = 1$$

is

$$-\frac{\sqrt{(1 - h)(hx^2 - h + h^2)}}{hx},$$

and if we replace y in

$$\frac{hxy}{hx^2 - y^2 - h}$$

by its value in

$$\frac{x^2}{1 - h} - \frac{y^2}{h} = 1,$$

we find the same value for the tangent of the angle of projection.

To get an idea of the appearance of the integral curves we need to show their behavior at a great distance from the origin.

In the case $h = \frac{1}{4}$ the boundary curve is an hyperbola and we know that we can project so as to hit the fixed point only from points inside this hyperbola. If in differential equation (6) we put $y/x = dy/dx = m$, considering y/x as the slope of the tangent for very large values of x we find

$$(h - m)^2 m^2 - 2hm^2 + (h - 1)m^2 = 0, \quad \text{or} \quad m^4 + m^2 = 0.$$

Thus we see that the slope of the tangents to the integral curves approaches zero as x increases, and that consequently these curves are asymptotic to the x -axis. Figure 3 shows some of the integral curves for $h = \frac{1}{4}$.

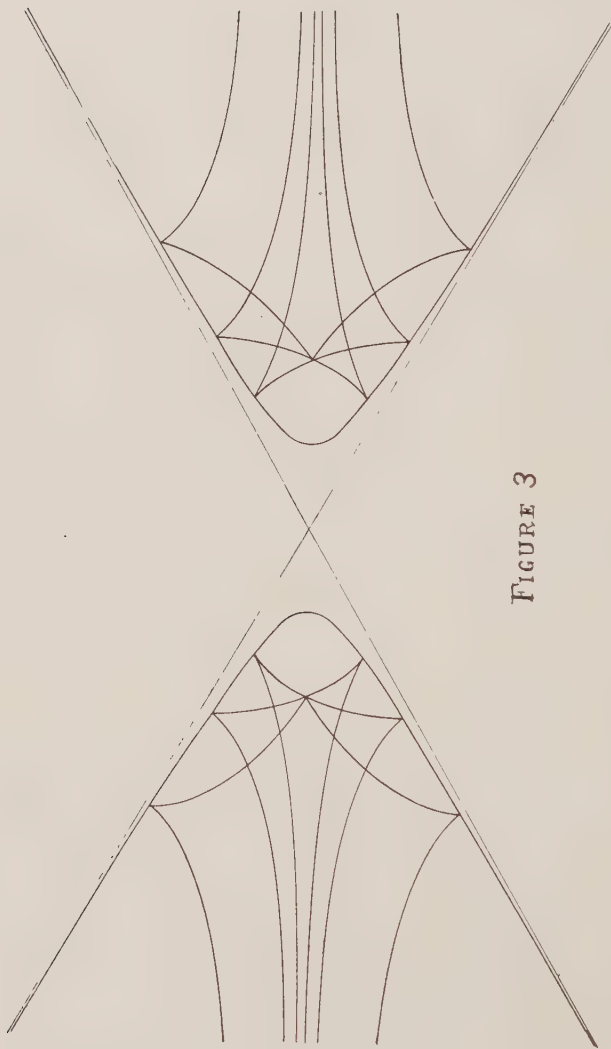


FIGURE 3

THE INVERSE SQUARE CASE

If a particle be projected with velocity v at an angle α in a field where the force varies inversely as the square of the distance the equations of motion are

$$(1) \quad \begin{aligned} \frac{d^2x}{dt^2} &= -\frac{\mu x}{r^3}, \\ \frac{d^2y}{dt^2} &= -\frac{\mu y}{r^3}. \end{aligned}$$

A first integration gives

$$(2) \quad x \frac{dy}{dt} - y \frac{dx}{dt} = h.$$

From the relation

$$r^2 \frac{d\theta}{dt} = h$$

we have from equations (1)

$$(3) \quad \begin{aligned} \frac{d^2x}{dt^2} &= -\frac{\mu}{h} \cos \theta \frac{d\theta}{dt}, \\ \frac{d^2y}{dt^2} &= -\frac{\mu}{h} \sin \theta \frac{d\theta}{dt}. \end{aligned}$$

Integrating these we get

$$(4) \quad \begin{aligned} \frac{dx}{dt} &= -\frac{\mu}{h} \sin \theta + A, \\ \frac{dy}{dt} &= \frac{\mu}{h} \cos \theta + B. \end{aligned}$$

When $x = x_0$ and $y = y_0$ we find

$$\begin{aligned} A &= v \cos \alpha + \frac{\mu}{h} \cdot \frac{y_0}{r_0}, \\ B &= v \sin \alpha - \frac{\mu}{h} \cdot \frac{x_0}{r_0}. \end{aligned}$$

Substituting the values from (4) in (2) we obtain for the equation of the path

$$(5) \quad \frac{\mu}{h} r + v(x \sin \alpha - y \cos \alpha) - \frac{\mu}{h} \left(\frac{x_0 x + y_0 y}{r_0} \right) = h.$$

From the relations

$$pv = h \quad \text{and} \quad p = x_0 \sin \alpha - y_0 \cos \alpha$$

we get finally

$$(6) \quad \mu r_0 r + v^2 r_0 (x \sin \alpha - y \cos \alpha) (x_0 \sin \alpha - y_0 \cos \alpha) \\ - \mu (x_0 x + y_0 y) = r_0^2 v^2 (x_0 \sin \alpha - y_0 \cos \alpha)^2.$$

If now we interchange the subscripts between x_0, x and y_0, y and consider α variable, putting $\tan \alpha = dy/dx$, we shall have a differential equation which will give the two directions of projection at any point in the plane in order to hit some fixed point which we are now calling (x_0, y_0) different from the point which we called (x_0, y_0) at the beginning. The differential equation is

$$(7) \quad [K(r_0 r - x_0 x - y_0 y) + r(x_0 x - x^2)] \left(\frac{dy}{dx} \right)^2 \\ - r(xy_0 + x_0 y - 2xy) \left(\frac{dy}{dx} \right) \\ + r(y_0 y - y^2) + K(r_0 r - x_0 x - y_0 y) = 0,$$

where $K = \mu/v^2$.

As has already been said, the transformation used in the other two cases discussed in this paper does not yield a sufficiently simple form to enable us to effect the solution of this differential equation.

BIOGRAPHICAL SKETCH

Allan Wilson Hobbs was born at Guilford College, N. C., July 12, 1885. He was graduated from that institution with the degree of Bachelor of Arts in 1907. After spending a year in further study at Haverford College he returned to Guilford College as Instructor in Mathematics. In 1911 he entered Johns Hopkins University as a graduate student in Mathematics, where he spent two years, returning to Guilford at the end of that time. In 1915 he entered Johns Hopkins again for another period of two years. He wishes to take this opportunity to express his appreciation of the courses of study pursued under the careful and stimulating direction of Professors Morley, Coble and Cohen in the Department of Mathematics and Professors Ames and Anderson in the Department of Physics.

